On the scattering of electromagnetic waves by small bodies

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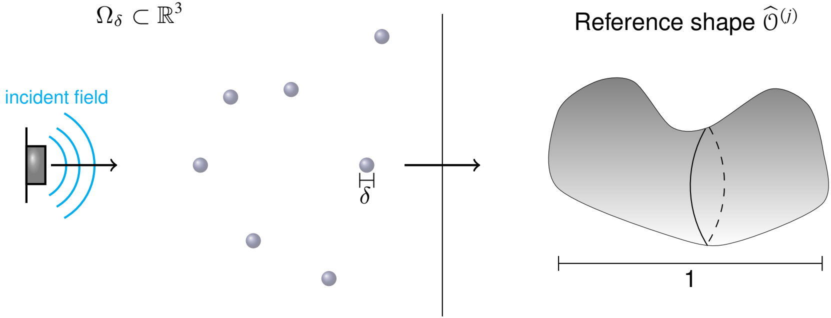
May 5, 2017

Scattering problems of electromagnetic waves by small inclusions in 3D



δ: small parameter

$\mathcal{O}_\delta^{(1)}, \ldots, \mathcal{O}_\delta^{(N)}$: inclusions

$\Omega_\delta = \mathbb{R}^3 \setminus \bigcup \overline{\mathcal{O}_\delta^{(j)}}$

$\Gamma_\delta^{(j)} = \partial \mathcal{O}_\delta^{(j)}$

$\Gamma_\delta = \bigcup \Gamma_\delta^{(j)}$

$\mathcal{O}_\delta^{(j)} = c_j + \delta \widehat{\mathcal{O}}^{(j)}$

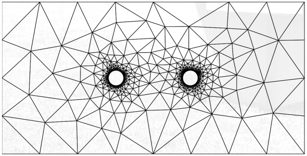
$\widehat{\mathcal{O}}^{(j)}$ Lipschitz-continuous

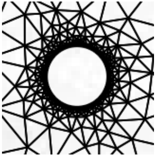
Asymptotic assumption

$$\delta \ll \lambda^{\mathrm{inc}}$$

Motivation – Applications and numerical difficulties

- Applications
  - Medical imaging: detecting small tumours
  - Non-destructive testing civil engineering: detecting impurities into concrete
- Numerical approximation difficulties
  - Infinite domain
  - Mesh refinement near bodies (pictures from G. Vial)





- Our approach
  - Reduced models derived thanks to asymptotic analysis: the radius δ is the small parameter
  - Integration in a Boundary Element library (unbounded domain)

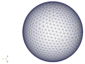
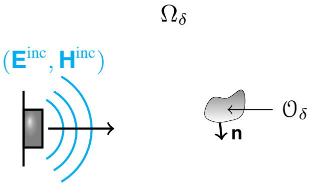


Figure: https://people.eecs.ku.edu/∼shontz/nsf_career_project.html

Non-exhaustive references – Scattering wave problems by small bodies

- **Electromagnetic waves**
  - H. Ammari, M. S. Vogelius, D. Volkov (2001)
    Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations
  - H. Ammari, H. Kang (2004)
    Reconstruction of small inhomogeneities from boundary measurements
- **Linear elasticity problems**
  - V. Bonnaillie-Noël, D. Brancherie, M. Dambrine, F. Hérau, S. Tordeux, G. Vial (2011)
    Multiscale expansion and numerical approximation for surface defects
- **Acoustic waves**
  - V. Mattesi (2014)
    Propagation des ondes dans un domaine comportant des petites hétérogénéités : modélisation asymptotique et calcul numérique
  - A. Bendali, P-H. Cocquet, S. Tordeux (2014)
    Approximation by multipoles of the multiple acoustic scattering by small obstacles and application to the Foldy theory of isotropic scattering

Scattering of electromagnetic waves by **one** small perfect condutor in 3D



$\mathcal{O}_\delta = \delta\widehat{\mathcal{O}}$: perfect conductor

$\Omega_\delta = \mathbb{R}^3 \setminus \overline{\mathcal{O}_\delta}$: exterior domain

$\Gamma_\delta = \partial\Omega_\delta$

κ: wave number

$\mathbf{E}^{\text{inc}}$, $\mathbf{H}^{\text{inc}}$: incident fields (given)

$\mathbf{E}_\delta^{\text{sca}}$, $\mathbf{H}_\delta^{\text{sca}}$: scattered fields

Time-harmonic Maxwell equations

$$\begin{cases} \textbf{curl}\ \mathbf{E}_\delta^{\text{sca}} - i\kappa\, \mathbf{H}_\delta^{\text{sca}} = 0 & \text{in } \Omega_\delta \\ \textbf{curl}\ \mathbf{H}_\delta^{\text{sca}} + i\kappa\, \mathbf{E}_\delta^{\text{sca}} = 0 & \text{in } \Omega_\delta \\ \mathbf{n} \times \mathbf{E}_\delta^{\text{sca}} = -\mathbf{n} \times \mathbf{E}^{\text{inc}} & \text{on } \Gamma_\delta \\ \mathbf{n} \cdot \mathbf{H}_\delta^{\text{sca}} = -\mathbf{n} \cdot \mathbf{H}^{\text{inc}} & \text{on } \Gamma_\delta \\ \lim\limits_{|\mathbf{x}| \to +\infty} |\mathbf{x}| \left(\mathbf{H}_\delta^{\text{sca}} \times \mathbf{e}_r - \mathbf{E}_\delta^{\text{sca}}\right) = 0 & \end{cases}$$

$$\begin{cases} \kappa^2 = \omega^2 \mu \left(\varepsilon + i\dfrac{\sigma}{\omega}\right) \\ \mathfrak{Im}(\kappa) \geq 0 \end{cases}$$

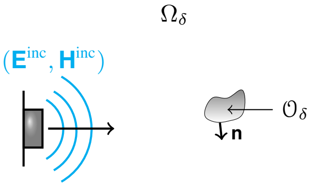
$\omega > 0$: frequency

$\mu > 0$: magnetic permeability

$\varepsilon > 0$: electric permittivity

$\sigma \geq 0$: electric conductivity

Scattering of electromagnetic waves by **one** small perfect condutor in 3D



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$$\begin{cases} \textbf{curl}\, \mathbf{E}^{\rm inc} - i\kappa\, \mathbf{H}^{\rm inc} = 0 & \text{in } \mathbb{R}^3 \\ \textbf{curl}\, \mathbf{H}^{\rm inc} + i\kappa\, \mathbf{E}^{\rm inc} = \mathbf{F} & \text{in } \mathbb{R}^3 \end{cases}$$

- **F** **smooth** source term
- $0 \notin \operatorname{supp} \mathbf{F}$

Guideline

Approximation method: Matched Asymptotic Expansions

"Formal" part

- Postulate asymptotic expansions
- Derive problems independent of δ
- Define a global approximation $\mathbf{E}_\delta^N$

Analysis part

- Modal representation of solutions
- Validation of reduced models

$$\left\| \mathbf{E}_\delta^N - \mathbf{E}_\delta^{\text{sca}} \right\|_X \stackrel{?}{=} \underset{\delta \to 0}{O}(\delta^N)$$

Numerical resolution method: Equivalent Source Problem

- Boundary obstacles considered as surface/ponctual source terms
- Implementation of Matched Asymptotic Expansions method
- Comparison with volumical methods

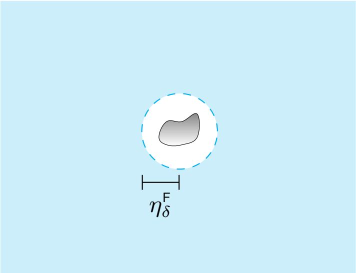
Outline

1. Method of matched asymptotic expansions
   - Asymptotic domain decomposition
   - Far field expansion
   - Near field expansion
   - Matching conditions
   - Global approximation
2. Modal decomposition of the asymptotics
   - Modal decomposition of the far field
   - Modal decomposition of the near field
   - Application of the matching conditions
   - Case of the sphere: analytical solutions
3. Conclusion and perspectives

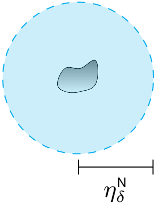
Outline

Asymptotic domain decomposition

Far field domain



Near field domain

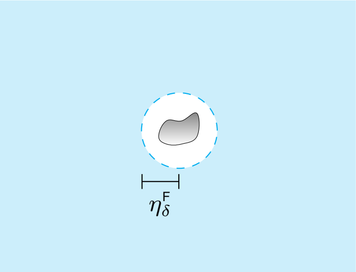


$$\lim_{\delta \to 0} \eta_\delta^{\mathsf{F}} = 0$$

$$\lim_{\delta \to 0} \frac{\eta_\delta^{\mathsf{N}}}{\delta} = +\infty$$

Asymptotic domain decomposition

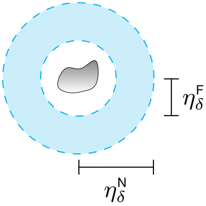
Far field domain



$$\lim_{\delta\to 0} \eta_\delta^{\mathsf{F}} = 0$$

$$\lim_{\delta\to 0} \frac{\eta_\delta^{\mathsf{N}}}{\delta} = +\infty$$

Matching area



Asymptotic assumption

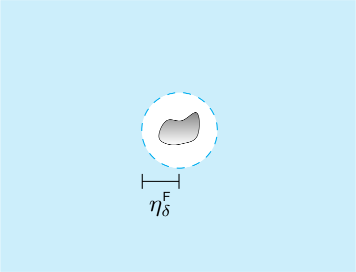
$$\delta \ll \eta_\delta^{\mathsf{F}} < \eta_\delta^{\mathsf{N}} \ll \lambda^{\mathrm{inc}}$$

A.M. Il'in (1991)

Matching of Asymptotic Expansions of Solutions of Boundary Value Problems

Asymptotic domain: far field

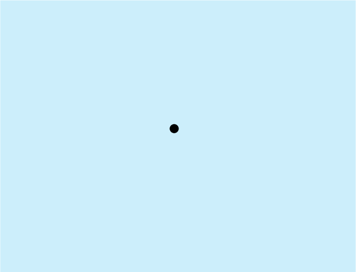
Far field domain



$$\mathbb{R}^3 \setminus \overline{\mathcal{B}(0, \eta_\delta^{\mathsf{F}})}$$

$$\delta \longrightarrow 0$$

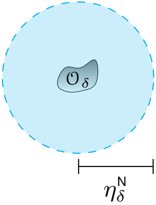
Asymptotic far field domain



$$\Omega^\star = \mathbb{R}^3 \setminus \{0\}$$

Asymptotic domain: near field

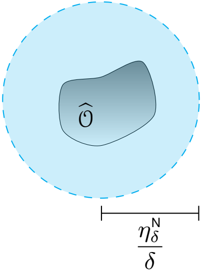
Near field domain



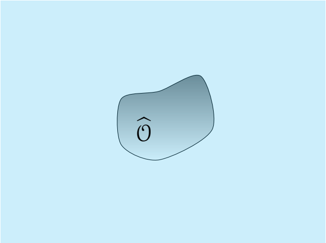
$$\mathcal{O}_\delta = \delta\widehat{\mathcal{O}}$$

$$\widehat{\mathbf{x}} = \frac{\mathbf{x}}{\delta}$$

Change of coordinates



$$\delta \to 0$$



Asymptotic near field domain

$$\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\widehat{\mathcal{O}}}$$

Far field expansion in $\Omega^\star = \mathbb{R}^3 \setminus \{0\}$

1. Postulate formal series

$$\mathbf{E}_\delta^{\mathrm{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \sum_{p=0}^{\infty} \delta^p \widetilde{\mathbf{E}}_p(\mathbf{x}) \qquad \mathbf{H}_\delta^{\mathrm{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \sum_{p=0}^{\infty} \delta^p \widetilde{\mathbf{H}}_p(\mathbf{x})$$

- $\widetilde{\mathbf{E}}_p$, $\widetilde{\mathbf{H}}_p$ defined in $\Omega^\star$
- $\widetilde{\mathbf{E}}_p$, $\widetilde{\mathbf{H}}_p$ independent of δ
- $\widetilde{\mathbf{E}}_p$, $\widetilde{\mathbf{H}}_p$ potentially singular at $\mathbf{x} = 0$

2. Derive formal problems : For $p \geq 0$, find $\widetilde{\mathbf{E}}_p$, $\widetilde{\mathbf{H}}_p \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}\,, \Omega^\star)$ s.t.

$$\begin{cases} \mathbf{curl}\, \widetilde{\mathbf{E}}_p - i\kappa\, \widetilde{\mathbf{H}}_p = 0 & \text{in } \Omega^\star \\ \mathbf{curl}\, \widetilde{\mathbf{H}}_p + i\kappa\, \widetilde{\mathbf{E}}_p = 0 & \text{in } \Omega^\star \end{cases}$$

+ Silver-Müller radiation condition

$$\lim_{|\mathbf{x}|\to+\infty} |\mathbf{x}| \left(\widetilde{\mathbf{H}}_p \times \mathbf{e}_r - \widetilde{\mathbf{E}}_p \right) = 0$$

- Ill-posed problems
- Missing information $\rightsquigarrow$ matching conditions
- Singular behavior at $\mathbf{x} = 0$

Far field expansion in $\Omega^\star = \mathbb{R}^3 \setminus \{0\}$

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- Missing information
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Near field expansion in $\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\widehat{\mathcal{O}}}$

1. Postulate formal series in fast variable $\widehat{\mathbf{x}} = \frac{\mathbf{x}}{\delta}$

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$$\mathbf{E}^{\mathrm{inc}}(\mathbf{x}) = \sum_{p=0}^{\infty} \delta^p \underbrace{\sum_{|\alpha|=p} \frac{1}{\alpha!} \mathrm{d}^\alpha \mathbf{E}^{\mathrm{inc}}(0) \cdot \widehat{\mathbf{x}}^\alpha}_{\widehat{\mathbf{E}}_p^{\mathrm{inc}}(\widehat{\mathbf{x}})}$$

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$(\widehat{\mathbf{E}}_{-1} = \widehat{\mathbf{H}}_{-1} = 0)$ + Boundary condition on $\widehat{\Gamma} = \partial\widehat{\Omega}$

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- Ill-posed problems
- Missing information $\rightsquigarrow$ matching conditions
- Increasing behavior at ∞

Some notations

Far field decomposition

$$\widetilde{\mathbf{E}}_p(\mathbf{x}) = \widetilde{\mathbf{E}}_p^{\text{reg}}(\mathbf{x}) + \widetilde{\mathbf{E}}_p^{\text{sing}}(\mathbf{x}) \qquad \mathbf{x} \in \Omega^\star$$

$$\widetilde{\mathbf{H}}_p(\mathbf{x}) = \widetilde{\mathbf{H}}_p^{\text{reg}}(\mathbf{x}) + \widetilde{\mathbf{H}}_p^{\text{sing}}(\mathbf{x}) \qquad \mathbf{x} \in \Omega^\star$$

- $\widetilde{\mathbf{E}}_p^{\text{reg}}, \widetilde{\mathbf{H}}_p^{\text{reg}}$: regular behavior at $\mathbf{x} = 0$ (**regular part** of the far field)
- $\widetilde{\mathbf{E}}_p^{\text{sing}}, \widetilde{\mathbf{H}}_p^{\text{sing}}$: singular behavior at $\mathbf{x} = 0$ (**singular part** of the far field)

Near field decomposition

$$\widehat{\mathbf{E}}_p(\widehat{\mathbf{x}}) = \widehat{\mathbf{E}}_p^{\text{reg}}(\widehat{\mathbf{x}}) + \widehat{\mathbf{E}}_p^{\text{sing}}(\widehat{\mathbf{x}}) \qquad \widehat{\mathbf{x}} \in \widehat{\Omega}$$

$$\widehat{\mathbf{H}}_p(\widehat{\mathbf{x}}) = \widehat{\mathbf{H}}_p^{\text{reg}}(\widehat{\mathbf{x}}) + \widehat{\mathbf{H}}_p^{\text{sing}}(\widehat{\mathbf{x}}) \qquad \widehat{\mathbf{x}} \in \widehat{\Omega}$$

- $\widehat{\mathbf{E}}_p^{\text{reg}}, \widehat{\mathbf{H}}_p^{\text{reg}}$: "decreasing" behavior at ∞ (**decreasing part** of the near field)
- $\widehat{\mathbf{E}}_p^{\text{sing}}, \widehat{\mathbf{H}}_p^{\text{sing}}$: increasing behavior at ∞ (**increasing part** of the near field)

Be careful. Weighted spaces

X. Claeys, PhD thesis (2008)
Analyse asymptotique et numérique de la diffraction d'ondes par des fils minces

Matching conditions

Find missing information

- Singular part of the far field $\longleftarrow$ Decreasing part of the near field
- Increasing part of the near field $\longleftarrow$ Regular part of the far field

Idea : develop $\widetilde{\mathbf{E}}_p$ close to $\mathbf{x} = 0$ and $\widehat{\mathbf{E}}_p$ near ∞

$$\widetilde{\mathbf{E}}_p(\mathbf{x}) \underset{r\to 0}{\sim} \sum_{\ell=-\infty}^{-1} r^\ell\, \widetilde{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) + \sum_{\ell=0}^{+\infty} r^\ell\, \widetilde{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) \qquad \mathbf{x} = (r,\theta,\varphi)$$

$$\widehat{\mathbf{E}}_p(\widehat{\mathbf{x}}) \underset{\mathsf{R}\to\infty}{\sim} \sum_{\ell=-\infty}^{-1} \mathsf{R}^\ell\, \widehat{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) + \sum_{\ell=0}^{+\infty} \mathsf{R}^\ell \widehat{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) \qquad \widehat{\mathbf{x}} = (\mathsf{R},\theta,\varphi) = \left(\frac{r}{\delta},\theta,\varphi\right)$$

Matching conditions

Find missing information

- Singular part of the far field $\longleftarrow$ Decreasing part of the near field
- Increasing part of the near field $\longleftarrow$ Regular part of the far field

Idea

- Far field expansion in $\mathscr{V}(0)$

$$\mathbf{E}_\delta^{\mathrm{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \sum_{p=-\infty}^{+\infty} \delta^p \left(\sum_{\ell=-\infty}^{-1} r^\ell\, \widetilde{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) + \sum_{\ell=0}^{+\infty} r^\ell\, \widetilde{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) \right)$$

- Near field expansion in $\mathscr{V}(\infty)$ with $\mathsf{R} = \frac{r}{\delta}$

$$\mathbf{E}_\delta^{\mathrm{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \sum_{p=-\infty}^{+\infty} \delta^p \left(\sum_{\ell=-\infty}^{-1} \mathsf{R}^\ell\, \widehat{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) + \sum_{\ell=0}^{+\infty} \mathsf{R}^\ell\, \widehat{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) \right)$$

Matching conditions

Find missing information

- Singular part of the far field $\longleftarrow$ Decreasing part of the near field
- Increasing part of the near field $\longleftarrow$ Regular part of the far field

Idea

- Far field expansion in $\mathscr{V}(0)$

$$\mathbf{E}_\delta^{\mathrm{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \sum_{p=-\infty}^{+\infty} \delta^p \left(\sum_{\ell=-\infty}^{+\infty} r^\ell \, \widetilde{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) \right)$$

- Near field expansion in $\mathscr{V}(\infty)$ with $\mathsf{R} = \frac{r}{\delta}$

$$\mathbf{E}_\delta^{\mathrm{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \sum_{p=-\infty}^{+\infty} \delta^p \left(\sum_{\ell=-\infty}^{+\infty} \mathsf{R}^\ell \, \widehat{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) \right)$$

- **Identification** of the two series

$$\begin{cases} \widetilde{\mathbf{E}}_\ell^{(p)} = \widehat{\mathbf{E}}_\ell^{(p+\ell)} & p \geq 0, \quad \ell = -p, \dots, -1 \\ \widehat{\mathbf{E}}_\ell^{(p)} = \widetilde{\mathbf{E}}_\ell^{(p-\ell)} & p \geq 0, \quad \ell = 0, \dots, p \end{cases}$$

$$\begin{cases} \widetilde{\mathbf{E}}_\ell^{(p)} = 0 & \ell < -p \\ \widehat{\mathbf{E}}_\ell^{(p)} = 0 & \ell > p \end{cases}$$

Matching conditions

- **Identification** of the two series

$$\begin{cases} \widetilde{\mathbf{E}}_\ell^{(p)} = \widehat{\mathbf{E}}_\ell^{(p+\ell)} & p \geq 0, \quad \ell = -p, \dots, -1 \\ \widehat{\mathbf{E}}_\ell^{(p)} = \widetilde{\mathbf{E}}_\ell^{(p-\ell)} & p \geq 0, \quad \ell = 0, \dots, p \end{cases} \qquad \begin{cases} \widetilde{\mathbf{E}}_\ell^{(p)} = 0 & \ell < -p \\ \widehat{\mathbf{E}}_\ell^{(p)} = 0 & \ell > p \end{cases}$$

- Term by term

$$\widetilde{\mathbf{E}}_p(\mathbf{x}) \underset{r\to 0}{\sim} \sum_{\ell=-p}^{-1} r^\ell\, \widehat{\mathbf{E}}_\ell^{(p+\ell)}(\theta,\varphi) + \sum_{\ell=0}^{+\infty} r^\ell\, \widetilde{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) \qquad \text{(far field)}$$

$$\widehat{\mathbf{E}}_p(\widehat{\mathbf{x}}) \underset{\mathrm{R}\to \infty}{\sim} \sum_{\ell=-\infty}^{-1} \mathrm{R}^\ell\, \widehat{\mathbf{E}}_\ell^{(p)}(\theta,\varphi) + \sum_{\ell=0}^{p} \mathrm{R}^\ell\, \widetilde{\mathbf{E}}_\ell^{(p-\ell)}(\theta,\varphi) \qquad \text{(near field)}$$

  - For $p = 0$, $\widetilde{\mathbf{E}}_0$ is **only** regular

Global approximation

Truncated series

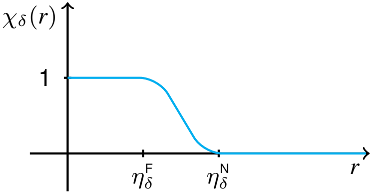
$$\widetilde{\mathbf{E}}_\delta^N(\mathbf{x}) = \sum_{p=0}^{N} \delta^p \, \widetilde{\mathbf{E}}_p(\mathbf{x}) \qquad \widetilde{\mathbf{H}}_\delta^P(\mathbf{x}) = \sum_{p=0}^{N} \delta^p \, \widetilde{\mathbf{H}}_p(\mathbf{x}) \qquad \text{(far fields)}$$

$$\widehat{\mathbf{E}}_\delta^N(\mathbf{x}) = \sum_{p=0}^{N} \delta^p \, \widehat{\mathbf{E}}_p(\widehat{\mathbf{x}}) \qquad \widehat{\mathbf{H}}_\delta^N(\mathbf{x}) = \sum_{p=0}^{N} \delta^p \, \widehat{\mathbf{H}}_p(\widehat{\mathbf{x}}) \qquad \text{(near fields)}$$

Global approximation

$$\mathbf{E}_\delta^P(\mathbf{x}) = \chi_\delta(|\mathbf{x}|)\, \widehat{\mathbf{E}}_\delta^P(\mathbf{x}) + \big(1 - \chi_\delta(|\mathbf{x}|)\big)\, \widetilde{\mathbf{E}}_\delta^P(\mathbf{x}) \qquad \mathbf{x} \in \Omega_\delta$$

$$\mathbf{H}_\delta^P(\mathbf{x}) = \chi_\delta(|\mathbf{x}|)\, \widehat{\mathbf{H}}_\delta^P(\mathbf{x}) + \big(1 - \chi_\delta(|\mathbf{x}|)\big)\, \widetilde{\mathbf{H}}_\delta^P(\mathbf{x}) \qquad \mathbf{x} \in \Omega_\delta$$



Consistance and stability Determine ϕ s.t. $\phi(N) \underset{N\to\infty}{\longrightarrow} +\infty$ and for any $N > 0$, $\rho > \rho_0$

$$\left\| \mathbf{E}_\delta^{\mathrm{sca}} - \mathbf{E}_\delta^N \right\|_{\mathbf{H}(\mathbf{curl}\,, \Omega_\delta \cap \mathcal{B}(0,\rho))} \le C_{N,\rho}^E \, \delta^{\phi(N)}$$

$$\left\| \mathbf{H}_\delta^{\mathrm{sca}} - \mathbf{H}_\delta^N \right\|_{\mathbf{H}(\mathbf{curl}\,, \Omega_\delta \cap \mathcal{B}(0,\rho))} \le C_{N,\rho}^H \, \delta^{\phi(N)}$$

Global approximation

Truncated series

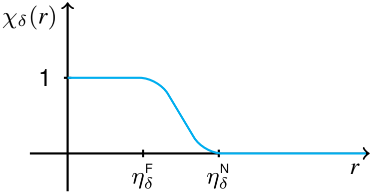
$$\widetilde{\mathbf{E}}_\delta^N(\mathbf{x}) = \sum_{p=0}^{N} \delta^p\, \widetilde{\mathbf{E}}_p(\mathbf{x}) \qquad \widetilde{\mathbf{H}}_\delta^P(\mathbf{x}) = \sum_{p=0}^{N} \delta^p\, \widetilde{\mathbf{H}}_p(\mathbf{x}) \qquad \text{(far fields)}$$

$$\widehat{\mathbf{E}}_\delta^N(\mathbf{x}) = \sum_{p=0}^{N} \delta^p\, \widehat{\mathbf{E}}_p(\widehat{\mathbf{x}}) \qquad \widehat{\mathbf{H}}_\delta^N(\mathbf{x}) = \sum_{p=0}^{N} \delta^p\, \widehat{\mathbf{H}}_p(\widehat{\mathbf{x}}) \qquad \text{(near fields)}$$

Global approximation

$$\mathbf{E}_\delta^P(\mathbf{x}) = \chi_\delta(|\mathbf{x}|)\, \widehat{\mathbf{E}}_\delta^P(\mathbf{x}) + \big(1 - \chi_\delta(|\mathbf{x}|)\big)\, \widetilde{\mathbf{E}}_\delta^P(\mathbf{x}) \qquad \mathbf{x} \in \Omega_\delta$$

$$\mathbf{H}_\delta^P(\mathbf{x}) = \chi_\delta(|\mathbf{x}|)\, \widehat{\mathbf{H}}_\delta^P(\mathbf{x}) + \big(1 - \chi_\delta(|\mathbf{x}|)\big)\, \widetilde{\mathbf{H}}_\delta^P(\mathbf{x}) \qquad \mathbf{x} \in \Omega_\delta$$



Consistance and stability Determine ϕ s.t. $\phi(N) \underset{N\to\infty}{\longrightarrow} +\infty$ and for any $N > 0$, $\rho > \rho_0$

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$$\left\| \mathbf{H}_\delta^{\mathrm{sca}} - \mathbf{H}_\delta^N \right\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_\delta \cap \mathcal{B}(0,\rho))} \le C_{N,\rho}^H\, \delta^{\phi(N)}$$

Outline

Modal decomposition of the far field of order 0

Find $\widetilde{\mathbf{E}}_0$ and $\widetilde{\mathbf{H}}_0 \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}\,, \mathbb{R}^3)$ s.t.

$$\begin{cases} \mathbf{curl}\,\widetilde{\mathbf{E}}_0 - i\kappa\,\widetilde{\mathbf{H}}_0 = 0 & \text{in } \mathbb{R}^3 \\ \mathbf{curl}\,\widetilde{\mathbf{H}}_0 + i\kappa\,\widetilde{\mathbf{E}}_0 = 0 & \text{in } \mathbb{R}^3 \\ \lim\limits_{|\mathbf{x}|\to\infty} |\mathbf{x}|\left(\widetilde{\mathbf{H}}_0 \times \mathbf{e}_r - \widetilde{\mathbf{E}}_0\right) = 0 & \end{cases}$$

- $\widetilde{\mathbf{E}}_0$, $\widetilde{\mathbf{H}}_0$ are **regular** (no singularity at $\mathbf{x} = 0$)
- $\widetilde{\mathbf{E}}_0$, $\widetilde{\mathbf{H}}_0$ satisfy S.M. $\Longrightarrow$ $\widetilde{\mathbf{E}}_0 = 0$ and $\widetilde{\mathbf{H}}_0 = 0$
- Far field expansion

$$\mathbf{E}_\delta^{\mathrm{sca}}(\mathbf{x}) \sim \sum_{p=1}^{\infty} \delta^p\, \widetilde{\mathbf{E}}_p(\mathbf{x})$$

Modal decomposition of the far field of order $p > 0$

Find $\widetilde{\mathbf{E}}_p$ and $\widetilde{\mathbf{H}}_p \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}\,, \Omega^\star)$ s.t.

$$\begin{cases} \mathbf{curl}\,\widetilde{\mathbf{E}}_p - i\kappa\,\widetilde{\mathbf{H}}_p = 0 & \text{in } \Omega^\star \\ \mathbf{curl}\,\widetilde{\mathbf{H}}_p + i\kappa\,\widetilde{\mathbf{E}}_p = 0 & \text{in } \Omega^\star \\ \lim\limits_{|\mathbf{x}|\to\infty} |\mathbf{x}|\left(\widetilde{\mathbf{H}}_p \times \mathbf{e}_r - \widetilde{\mathbf{E}}_p\right) = 0 & \end{cases}$$

- $\widetilde{\mathbf{E}}_p$, $\widetilde{\mathbf{H}}_p$ are **singular** at $\mathbf{x} = 0$

Modal decomposition of the far field of order $p > 0$

Find $\widetilde{\mathbf{E}}_p$ and $\widetilde{\mathbf{H}}_p \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}\,, \Omega^\star)$ s.t.

$$\begin{cases} \mathbf{curl}\ \widetilde{\mathbf{E}}_p - i\kappa\, \widetilde{\mathbf{H}}_p = 0 & \text{in } \Omega^\star \\ \mathbf{curl}\ \widetilde{\mathbf{H}}_p + i\kappa\, \widetilde{\mathbf{E}}_p = 0 & \text{in } \Omega^\star \\ \lim\limits_{|\mathbf{x}|\to\infty} |\mathbf{x}| \left(\widetilde{\mathbf{H}}_p \times \mathbf{e}_r - \widetilde{\mathbf{E}}_p \right) = 0 & \end{cases}$$

- $\widetilde{\mathbf{E}}_p$, $\widetilde{\mathbf{H}}_p$ are **singular** at $\mathbf{x} = 0$
- Modal decomposition (singular at 0)

$$\widetilde{\mathbf{E}}_p(\mathbf{x}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left\{ \widetilde{\beta}_{n,m}^{(p),\mathsf{E}}\, \mathbf{curl}\ \left[h_n^{(1)}(\kappa|\mathbf{x}|) Y_{n,m}(\widehat{x})\mathbf{x} \right] + \frac{\widetilde{\beta}_{n,m}^{(p),\mathsf{H}}}{i\kappa} \mathbf{curl}\ \mathbf{curl}\ \left[h_n^{(1)}(\kappa|\mathbf{x}|) Y_{n,m}(\widehat{x})\mathbf{x} \right] \right\}$$

  - **Finite number** of $\widetilde{\beta}_{n,m}^{(p),\mathsf{E}}$, $\widetilde{\beta}_{n,m}^{(p),\mathsf{H}}$ non-equal to 0
  - $\widetilde{\beta}_{n,m}^{(p),\mathsf{E}}$, $\widetilde{\beta}_{n,m}^{(p),\mathsf{H}}$ are given by **matching conditions**
  - $h_n^{(1)}$: spherical Hankel function of first kind

Modal decomposition of the far field of order $p > 0$

Find $\widetilde{\mathbf{E}}_p$ and $\widetilde{\mathbf{H}}_p \in \mathbf{H}_{\rm loc}(\mathbf{curl}\,, \Omega^\star)$ s.t.

$$\begin{cases} \mathbf{curl}\, \widetilde{\mathbf{E}}_p - i\kappa\, \widetilde{\mathbf{H}}_p = 0 & \text{in } \Omega^\star \\ \mathbf{curl}\, \widetilde{\mathbf{H}}_p + i\kappa\, \widetilde{\mathbf{E}}_p = 0 & \text{in } \Omega^\star \\ \lim\limits_{|\mathbf{x}|\to\infty} |\mathbf{x}| \left(\widetilde{\mathbf{H}}_p \times \mathbf{e}_r - \widetilde{\mathbf{E}}_p \right) = 0 & \end{cases}$$

- $\widetilde{\mathbf{E}}_p$, $\widetilde{\mathbf{H}}_p$ are **singular** at $\mathbf{x} = 0$
- Modal decomposition (singular at 0)

$$\widetilde{\mathbf{E}}_p(\mathbf{x}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \Bigg\{ \widetilde{\beta}_{n,m}^{(p),\mathsf{E}} h_n^{(1)}(\kappa r) \mathbf{curl}_{\mathbb{S}^2} Y_{n,m}(\theta,\varphi) + \frac{\widetilde{\beta}_{n,m}^{(p),\mathsf{H}}}{i\kappa r} \left(\left[h_n^{(1)}(\kappa r) + \kappa r\, h_n^{(1)\prime}(\kappa r) \right] \nabla_{\mathbb{S}^2} Y_{n,m}(\theta,\varphi) + n(n+1) h_n^{(1)}(\kappa r) Y_{n,m}(\theta,\varphi) \mathbf{e}_r \right) \Bigg\}$$

- **Finite number** of $\widetilde{\beta}_{n,m}^{(p),\mathsf{E}}$, $\widetilde{\beta}_{n,m}^{(p),\mathsf{H}}$ non-equal to 0
- $\widetilde{\beta}_{n,m}^{(p),\mathsf{E}}$, $\widetilde{\beta}_{n,m}^{(p),\mathsf{H}}$ are given by **matching conditions**

$\rightarrow$ **Behavior at 0 ?**

Behavior of the far field in a neighbourhood of 0

Expansion of spherical Hankel function (1st)

$$h_n^{(1)}(z) = \sum_{\ell \geq -n-1} h_{n,\ell}\, z^{\ell} \qquad \Big| \qquad h_{n,\ell} = 0 \text{ iff } (\ell - n \text{ even and } \ell < n) \text{ or } (\ell < -n-1)$$

Expansion of the far field close to 0

$$\widetilde{\mathbf{E}}_{p>0}(\mathbf{x}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \Bigg\{ \widetilde{\beta}_{n,m}^{(p),\mathsf{E}} \sum_{\ell \geq -n-1} \Big\{ (\kappa r)^{\ell}\, h_{n,\ell}\, \mathbf{curl}_{\mathbb{S}^2} Y_{n,m}(\theta,\varphi) \Big\} + \widetilde{\beta}_{n,m}^{(p),\mathsf{H}} \sum_{\ell \geq -n-2} \Big\{ (\kappa r)^{\ell}\, \frac{h_{n,\ell+1}}{i} \big[(\ell+2) \nabla_{\mathbb{S}^2} Y_{n,m}(\theta,\varphi) + n(n+1) Y_{n,m}(\theta,\varphi) \mathbf{e}_r \big] \Big\} \Bigg\}$$

No **regular** part for the far field $\Longrightarrow$ No **increasing** part for the near field

Behavior of the far field in a neighbourhood of 0

Expansion of spherical Hankel function (1st)

$$h_n^{(1)}(z) = \sum_{\ell \geq -n-1} h_{n,\ell}\, z^{\ell} \qquad \Big| \quad h_{n,\ell} = 0 \text{ iff } (\ell - n \text{ even and } \ell < n) \text{ or } (\ell < -n-1)$$

Expansion of the far field close to 0

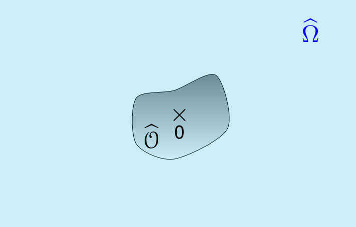
$$\widetilde{\mathbf{E}}_{p>0}(\mathbf{x}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left\{ \widetilde{\beta}_{n,m}^{(p),\mathsf{E}} \sum_{\ell \geq -n-1} \left\{ (\kappa r)^{\ell}\, h_{n,\ell}\, \mathbf{curl}_{\mathbb{S}^2} Y_{n,m}(\theta,\varphi) \right\} + \widetilde{\beta}_{n,m}^{(p),\mathsf{H}} \sum_{\ell \geq -n-2} \left\{ (\kappa r)^{\ell}\, \frac{h_{n,\ell+1}}{i} \left[(\ell+2) \nabla_{\mathbb{S}^2} Y_{n,m}(\theta,\varphi) + n(n+1) Y_{n,m}(\theta,\varphi)\mathbf{e}_r \right] \right\} \right\}$$

No **regular** part for the far field $\Longrightarrow$ No **increasing** part for the near field

Modal decomposition of the near field

Find $(\widehat{\mathbf{E}}_p, \widehat{\mathbf{H}}_p)_{p\geq 0} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}\,, \widehat{\Omega})$ s.t.

$$\begin{cases} \mathbf{curl}\, \widehat{\mathbf{E}}_p = i\kappa\, \widehat{\mathbf{H}}_{p-1} & \text{in } \widehat{\Omega} \\ \mathbf{curl}\, \widehat{\mathbf{H}}_p = -i\kappa\, \widehat{\mathbf{E}}_{p-1} & \text{in } \widehat{\Omega} \\ \mathbf{n} \times \widehat{\mathbf{E}}_p = -\mathbf{n} \times \widehat{\mathbf{E}}_p^{\text{inc}} \;(*) & \text{on } \widehat{\Gamma} \\ \mathbf{n} \cdot \widehat{\mathbf{H}}_p = -\mathbf{n} \cdot \widehat{\mathbf{H}}_p^{\text{inc}} & \text{on } \widehat{\Gamma} \end{cases}$$



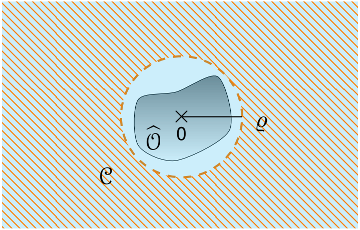
$$(*) \quad \widehat{\mathbf{E}}_p^{\text{inc}}(\widehat{\mathbf{x}}) = \sum_{|\alpha|=p} \frac{1}{\alpha!} \mathrm{d}^\alpha \mathbf{E}^{\text{inc}}(0) \cdot \widehat{\mathbf{x}}^\alpha$$

- No **increasing** part for the near field
- Matching conditions $-$ **separation of variables**
- **“Decreasing”** part$-$ by induction with **integral equations**

Modal decomposition of the near field

$$\left\{\begin{array}{lll} \textbf{curl}\, \widehat{\mathbf{E}}_{p,0} & = 0 & \text{in } \mathcal{C} \\ \textbf{curl}\, \widehat{\mathbf{H}}_{p,0} & = 0 & \text{in } \mathcal{C} \\ \operatorname{div} \widehat{\mathbf{E}}_{p,0} & = 0 & \text{in } \mathcal{C} \\ \operatorname{div} \widehat{\mathbf{H}}_{p,0} & = 0 & \text{in } \mathcal{C} \end{array}\right.$$

$$\left\{\begin{array}{lll} \textbf{curl}\, \widehat{\mathbf{E}}_{p,\ell} & = i\kappa\, \widehat{\mathbf{H}}_{p-1,\ell-1} & \text{in } \mathcal{C} \\ \textbf{curl}\, \widehat{\mathbf{H}}_{p,\ell} & = -i\kappa\, \widehat{\mathbf{E}}_{p-1,\ell-1} & \text{in } \mathcal{C} \end{array}\right.$$



$$\mathcal{C} = \left\{\widehat{\mathbf{x}} \in \mathbb{R}^3,\ |\widehat{\mathbf{x}}| > \varrho\right\}$$

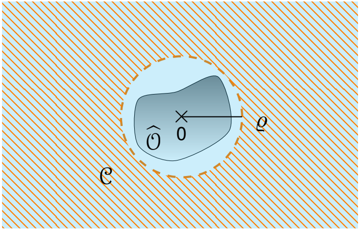
Near field expansions into $\mathcal{C}$

$$\widehat{\mathbf{E}}_p(\widehat{\mathbf{x}}) = \underbrace{\widehat{\mathbf{E}}_{p,0}(\widehat{\mathbf{x}})}_{\text{HEAD}} + \underbrace{\sum_{\ell=1}^{p} \widehat{\mathbf{E}}_{p,\ell}(\widehat{\mathbf{x}})}_{\text{SHADOWS}} \qquad \widehat{\mathbf{H}}_p(\widehat{\mathbf{x}}) = \underbrace{\widehat{\mathbf{H}}_{p,0}(\widehat{\mathbf{x}})}_{\text{HEAD}} + \underbrace{\sum_{\ell=1}^{p} \widehat{\mathbf{H}}_{p,\ell}(\widehat{\mathbf{x}})}_{\text{SHADOWS}}$$

- HEADs : solution of homogeneous Maxwell **static** equations
- SHADOWS : particular solution of **nested** equations

Modal decomposition of the near field

$$\begin{cases} \textbf{curl}\, \widehat{\mathbf{E}}_{p,0} = 0 & \text{in } \mathcal{C} \\ \textbf{curl}\, \widehat{\mathbf{H}}_{p,0} = 0 & \text{in } \mathcal{C} \\ \operatorname{div} \widehat{\mathbf{E}}_{p,0} = 0 & \text{in } \mathcal{C} \\ \operatorname{div} \widehat{\mathbf{H}}_{p,0} = 0 & \text{in } \mathcal{C} \end{cases}$$



$$\mathcal{C} = \left\{ \widehat{\mathbf{x}} \in \mathbb{R}^3,\ |\widehat{\mathbf{x}}| > \varrho \right\}$$

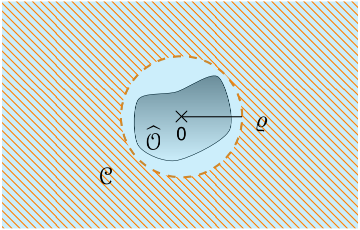
Head expansions into $\mathcal{C}$

$$\widehat{\mathbf{E}}_{p,0} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (\kappa \mathsf{R})^{-n-2}\, \widehat{\alpha}_{n,m}^{(p),\mathsf{H}}\, \frac{h_{n,-n-1}}{i} \left[-n \nabla_{\mathbb{S}^2} Y_{n,m} + n(n+1)\, Y_{n,m} \mathbf{e}_r \right]$$

$$\widehat{\mathbf{H}}_{p,0} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} (\kappa \mathsf{R})^{-n-2}\, \widehat{\alpha}_{n,m}^{(p),\mathsf{E}}\, \frac{h_{n,-n-1}}{i} \left[-n \nabla_{\mathbb{S}^2} Y_{n,m} + n(n+1)\, Y_{n,m} \mathbf{e}_r \right]$$

Modal decomposition of the near field

$$\begin{cases} \textbf{curl}\, \widehat{\mathbf{E}}_{p,\ell} = i\kappa\, \widehat{\mathbf{H}}_{p-1,\ell-1} & \text{in } \mathcal{C} \\ \textbf{curl}\, \widehat{\mathbf{H}}_{p,\ell} = -i\kappa\, \widehat{\mathbf{E}}_{p-1,\ell-1} & \text{in } \mathcal{C} \end{cases}$$



Shadow expansions into $\mathcal{C}$

$$\mathcal{C} = \{\widehat{\mathbf{x}} \in \mathbb{R}^3,\ |\widehat{\mathbf{x}}| > \varrho\}$$

$$\widehat{\mathbf{E}}_{p,\ell} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} (\kappa \mathrm{R})^{\ell-n-2} \Big\{ \widehat{\alpha}_{n,m}^{(p-\ell),\mathrm{E}} h_{n,\ell-n-2}\, \textbf{curl}_{\mathbb{S}^2} Y_{n,m} + \widehat{\alpha}_{n,m}^{(p-\ell),\mathrm{H}} \frac{h_{n,\ell-n-1}}{i} \Big[(\ell-n)\nabla_{\mathbb{S}^2} Y_{n,m} + n(n+1) Y_{n,m} \mathbf{e}_r\Big] \Big\}$$

$$\widehat{\mathbf{H}}_{p,\ell} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} (\kappa \mathrm{R})^{\ell-n-2} \Big\{ -\widehat{\alpha}_{n,m}^{(p-\ell),\mathrm{H}} h_{n,\ell-n-2}\, \textbf{curl}_{\mathbb{S}^2} Y_{n,m} + \widehat{\alpha}_{n,m}^{(p-\ell),\mathrm{E}} \frac{h_{n,\ell-n-1}}{i} \Big[(\ell-n)\nabla_{\mathbb{S}^2} Y_{n,m} + n(n+1) Y_{n,m} \mathbf{e}_r\Big] \Big\}$$

Application of the matching conditions

- **Far field expansion**

$$\widetilde{\mathbf{E}}_p = \sum_{n=1}^{p-2} \sum_{m=-n}^{n} \Big\{ \widehat{\alpha}_{n,m}^{(p-n-2),\mathsf{E}} h_n^{(1)}(\kappa r)\,\mathbf{curl}_{\mathbb{S}^2} Y_{n,m} + \frac{\widehat{\alpha}_{n,m}^{(p-n-2),\mathsf{H}}}{i\kappa r} \Big(\big[h_n^{(1)}(\kappa r) + \kappa r\, h_n^{(1)\prime}(\kappa r)\big] \nabla_{\mathbb{S}^2} Y_{n,m} + n(n+1)\, h_n^{(1)}(\kappa r) Y_{n,m} \mathbf{e}_r \Big) \Big\}$$

- *Resolution of the near field problem?*

$\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}}$

✓	$\widetilde{\mathbf{E}}_0, \widetilde{\mathbf{H}}_0$	$\widehat{\mathbf{E}}_0, \widehat{\mathbf{H}}_0$
✓	$\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{H}}_1$	$\widehat{\mathbf{E}}_1, \widehat{\mathbf{H}}_1$
✓	$\widetilde{\mathbf{E}}_2, \widetilde{\mathbf{H}}_2$	$\widehat{\mathbf{E}}_2, \widehat{\mathbf{H}}_2$
	$\widetilde{\mathbf{E}}_3, \widetilde{\mathbf{H}}_3$	$\widehat{\mathbf{E}}_3, \widehat{\mathbf{H}}_3$
	$\widetilde{\mathbf{E}}_4, \widetilde{\mathbf{H}}_4$	$\widehat{\mathbf{E}}_4, \widehat{\mathbf{H}}_4$
	$\widetilde{\mathbf{E}}_5, \widetilde{\mathbf{H}}_5$	$\cdots$
	$\cdots$	

Application of the matching conditions

- **Far field expansion**

$$\widetilde{\mathbf{E}}_p = \sum_{n=1}^{p-2} \sum_{m=-n}^{n} \Big\{ \widehat{\alpha}_{n,m}^{(p-n-2),\mathsf{E}} h_n^{(1)}(\kappa r)\,\mathbf{curl}_{\mathbb{S}^2} Y_{n,m} + \frac{\widehat{\alpha}_{n,m}^{(p-n-2),\mathsf{H}}}{i\kappa r}\Big(\big[h_n^{(1)}(\kappa r) + \kappa r\, h_n^{(1)\prime}(\kappa r)\big]\nabla_{\mathbb{S}^2} Y_{n,m} + n(n+1)\, h_n^{(1)}(\kappa r) Y_{n,m}\mathbf{e}_r\Big)\Big\}$$

- *Resolution of the near field problem?*

$$\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}}$$

✓	$\widetilde{\mathbf{E}}_0, \widetilde{\mathbf{H}}_0$	$\boxed{\widehat{\mathbf{E}}_0, \widehat{\mathbf{H}}_0}$	✓
✓	$\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{H}}_1$	$\widehat{\mathbf{E}}_1, \widehat{\mathbf{H}}_1$	✓
✓	$\widetilde{\mathbf{E}}_2, \widetilde{\mathbf{H}}_2$	$\widehat{\mathbf{E}}_2, \widehat{\mathbf{H}}_2$	✓
	$\widetilde{\mathbf{E}}_3, \widetilde{\mathbf{H}}_3$	$\widehat{\mathbf{E}}_3, \widehat{\mathbf{H}}_3$	✓
	$\widetilde{\mathbf{E}}_4, \widetilde{\mathbf{H}}_4$	$\widehat{\mathbf{E}}_4, \widehat{\mathbf{H}}_4$	✓
	$\widetilde{\mathbf{E}}_5, \widetilde{\mathbf{H}}_5$	$\cdots$	✓
	$\cdots$		

Application of the matching conditions

- **Far field expansion**

$$\widetilde{\mathbf{E}}_p = \sum_{n=1}^{p-2} \sum_{m=-n}^{n} \Bigg\{ \widehat{\alpha}_{n,m}^{(p-n-2),\mathsf{E}} h_n^{(1)}(\kappa r)\,\mathbf{curl}_{\mathbb{S}^2} Y_{n,m} + \frac{\widehat{\alpha}_{n,m}^{(p-n-2),\mathsf{H}}}{i\kappa r} \Big(\big[h_n^{(1)}(\kappa r) + \kappa r\, h_n^{(1)\prime}(\kappa r)\big] \nabla_{\mathbb{S}^2} Y_{n,m} + n(n+1)\, h_n^{(1)}(\kappa r) Y_{n,m} \mathbf{e}_r \Big) \Bigg\}$$

- *Resolution of the near field problem?*

$\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}}$

✓	$\widetilde{\mathbf{E}}_0, \widetilde{\mathbf{H}}_0$	$\boxed{\widehat{\mathbf{E}}_0, \widehat{\mathbf{H}}_0}$	✓
✓	$\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{H}}_1$	$\boxed{\widehat{\mathbf{E}}_1, \widehat{\mathbf{H}}_1}$	✓
✓	$\widetilde{\mathbf{E}}_2, \widetilde{\mathbf{H}}_2$	$\widehat{\mathbf{E}}_2, \widehat{\mathbf{H}}_2$	✓
✓	$\widetilde{\mathbf{E}}_3, \widetilde{\mathbf{H}}_3$	$\widehat{\mathbf{E}}_3, \widehat{\mathbf{H}}_3$	✓
	$\widetilde{\mathbf{E}}_4, \widetilde{\mathbf{H}}_4$	$\widehat{\mathbf{E}}_4, \widehat{\mathbf{H}}_4$	✓
	$\widetilde{\mathbf{E}}_5, \widetilde{\mathbf{H}}_5$	$\dots$	✓
	$\dots$		

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- *Resolution of the near field problem?*

$\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}}$

✓	$\widetilde{\mathbf{E}}_0, \widetilde{\mathbf{H}}_0$	$\boxed{\widehat{\mathbf{E}}_0, \widehat{\mathbf{H}}_0}$	✓
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✓	$\widetilde{\mathbf{E}}_2, \widetilde{\mathbf{H}}_2$	$\boxed{\widehat{\mathbf{E}}_2, \widehat{\mathbf{H}}_2}$	✓
✓	$\widetilde{\mathbf{E}}_3, \widetilde{\mathbf{H}}_3$	$\widehat{\mathbf{E}}_3, \widehat{\mathbf{H}}_3$	✓
✓	$\widetilde{\mathbf{E}}_4, \widetilde{\mathbf{H}}_4$	$\widehat{\mathbf{E}}_4, \widehat{\mathbf{H}}_4$	✓
	$\widetilde{\mathbf{E}}_5, \widetilde{\mathbf{H}}_5$	$\cdots$	✓
	$\cdots$		

Application of the matching conditions

- **Far field expansion**

$$\widetilde{\mathbf{E}}_p = \sum_{n=1}^{p-2} \sum_{m=-n}^{n} \Bigg\{ \widehat{\alpha}_{n,m}^{(p-n-2),\mathsf{E}} h_n^{(1)}(\kappa r)\,\mathbf{curl}_{\mathbb{S}^2} Y_{n,m} + \frac{\widehat{\alpha}_{n,m}^{(p-n-2),\mathsf{H}}}{i\kappa r} \Big(\big[h_n^{(1)}(\kappa r) + \kappa r\, h_n^{(1)\prime}(\kappa r)\big] \nabla_{\mathbb{S}^2} Y_{n,m} + n(n+1)\, h_n^{(1)}(\kappa r) Y_{n,m} \mathbf{e}_r \Big) \Bigg\}$$

- *Resolution of the near field problem?*

Etcetera ...

$\mathbf{E}^{\rm inc}, \mathbf{H}^{\rm inc}$

✓	$\widetilde{\mathbf{E}}_0, \widetilde{\mathbf{H}}_0$	$\widehat{\mathbf{E}}_0, \widehat{\mathbf{H}}_0$	✓
✓	$\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{H}}_1$	$\widehat{\mathbf{E}}_1, \widehat{\mathbf{H}}_1$	✓
✓	$\widetilde{\mathbf{E}}_2, \widetilde{\mathbf{H}}_2$	$\widehat{\mathbf{E}}_2, \widehat{\mathbf{H}}_2$	✓
✓	$\widetilde{\mathbf{E}}_3, \widetilde{\mathbf{H}}_3$	$\widehat{\mathbf{E}}_3, \widehat{\mathbf{H}}_3$	✓
✓	$\widetilde{\mathbf{E}}_4, \widetilde{\mathbf{H}}_4$	$\widehat{\mathbf{E}}_4, \widehat{\mathbf{H}}_4$	✓
✓	$\widetilde{\mathbf{E}}_5, \widetilde{\mathbf{H}}_5$	$\cdots$	✓
	$\cdots$		

Explicit computation in the case of the sphere $\mathcal{O}_\delta = \mathcal{B}(0,\delta)$

- **Local approximations**

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \delta^3\,\widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4\,\widetilde{\mathbf{E}}_4(\mathbf{x}) + \underset{\delta\to 0}{O}(\delta^5)$$

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \widehat{\mathbf{E}}_0(\widehat{\mathbf{x}}) + \delta\,\widehat{\mathbf{E}}_1(\widehat{\mathbf{x}}) + \underset{\delta\to 0}{O}(\delta^2)$$

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- **Expression of $\widetilde{\mathbf{E}}_3$ (matching conditions)**

$$\widetilde{\mathbf{E}}_3 = \sum_{m=-1}^{1} \Bigg\{ \widehat{\alpha}_{1,m}^{(0),\mathsf{E}} h_1^{(1)}(\kappa r)\,\mathbf{curl}_{\mathbb{S}^2} Y_{1,m} + \frac{\widehat{\alpha}_{1,m}^{(0),\mathsf{H}}}{i\kappa r}\left(\left[h_1^{(1)}(\kappa r) + \kappa r\, h_1^{(1)\prime}(\kappa r)\right]\nabla_{\mathbb{S}^2} Y_{1,m} + 2\,h_1^{(1)}(\kappa r) Y_{1,m}\mathbf{e}_r\right)\Bigg\}$$

Explicit computation in the case of the sphere $\mathcal{O}_\delta = \mathcal{B}(0,\delta)$

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$$\mathbf{E}_\delta^{\mathrm{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \delta^3\, \widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4\, \widetilde{\mathbf{E}}_4(\mathbf{x}) + \underset{\delta\to 0}{O}(\delta^5)$$

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- **Modal representation of $(\widehat{\mathbf{E}}_0, \widehat{\mathbf{H}}_0)$**

$$\widehat{\mathbf{E}}_0(\mathbf{X}) = (\kappa \mathsf{R})^{-3}\, \frac{h_{1,-2}}{i} \sum_{m=-1}^{1} \widehat{\alpha}_{1,m}^{(0),\mathsf{H}} \left[-\nabla_{\mathcal{S}^2} Y_{1,m} + 2\, Y_{1,m} \mathbf{e}_r \right] + \underset{\mathsf{R}\to\infty}{O}(\mathsf{R}^{-4})$$

$$\widehat{\mathbf{H}}_0(\mathbf{X}) = (\kappa \mathsf{R})^{-3}\, \frac{h_{1,-2}}{i} \sum_{m=-1}^{1} \widehat{\alpha}_{1,m}^{(0),\mathsf{E}} \left[-\nabla_{\mathcal{S}^2} Y_{1,m} + 2\, Y_{1,m} \mathbf{e}_r \right] + \underset{\mathsf{R}\to\infty}{O}(\mathsf{R}^{-4})$$

Boundary conditions on $\widehat{\Gamma} = \mathcal{S}^2 \Longrightarrow$ Unicity of $\widehat{\alpha}_{1,m}^{(0),\mathsf{H}}$ and $\widehat{\alpha}_{1,m}^{(0),\mathsf{E}}$

Explicit computation in the case of the sphere $\mathcal{O}_\delta = \mathcal{B}(0,\delta)$

- **Local approximations**

$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \delta^3\, \widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4\, \widetilde{\mathbf{E}}_4(\mathbf{x}) + \underset{\delta\to 0}{O}(\delta^5)$$

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- **Expression of $\widetilde{\mathbf{E}}_4$ (matching conditions)**

$$\widetilde{\mathbf{E}}_4 = \sum_{m=-1}^{1} \left\{ \widehat{\alpha}_{1,m}^{(1),\mathsf{E}} h_1^{(1)}(\kappa r)\,\mathbf{curl}_{\mathbb{S}^2} Y_{1,m} + \frac{\widehat{\alpha}_{1,m}^{(1),\mathsf{H}}}{i\kappa r}\left(\left[h_1^{(1)}(\kappa r) + \kappa r\, h_1^{(1)\prime}(\kappa r)\right]\nabla_{\mathbb{S}^2} Y_{1,m} + 2\, h_1^{(1)}(\kappa r) Y_{1,m}\mathbf{e}_r\right)\right\}$$
$$+ \sum_{m=-2}^{2} \left\{ \widehat{\alpha}_{2,m}^{(0),\mathsf{E}} h_2^{(1)}(\kappa r)\,\mathbf{curl}_{\mathbb{S}^2} Y_{2,m} + \frac{\widehat{\alpha}_{2,m}^{(0),\mathsf{H}}}{i\kappa r}\left(\left[h_2^{(1)}(\kappa r) + \kappa r\, h_2^{(1)\prime}(\kappa r)\right]\nabla_{\mathbb{S}^2} Y_{2,m} + 6\, h_2^{(1)}(\kappa r) Y_{2,m}\mathbf{e}_r\right)\right\}$$

Explicit computation in the case of the sphere $\mathcal{O}_\delta = \mathcal{B}(0, \delta)$

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$$\mathbf{E}_\delta^{\text{sca}}(\mathbf{x}) \underset{\delta\to 0}{\sim} \delta^3\, \widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4\, \widetilde{\mathbf{E}}_4(\mathbf{x}) + \underset{\delta\to 0}{O}(\delta^5)$$

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- **Modal representation of $(\widehat{\mathbf{E}}_1, \widehat{\mathbf{H}}_1)$**

$$\widehat{\mathbf{E}}_1(\mathbf{X}) = (\kappa \mathsf{R})^{-3}\, \frac{h_{1,-2}}{i} \sum_{m=-1}^{1} \widehat{\alpha}_{1,m}^{(1),\mathsf{H}} \left[-\nabla_{\mathcal{S}^2} Y_{1,m} + 2\, Y_{1,m}\mathbf{e}_r\right] + \underset{\mathsf{R}\to\infty}{O}(\mathsf{R}^{-4})$$

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Boundary conditions on $\widehat{\Gamma} = \mathcal{S}^2 \Longrightarrow$ Unicity of $\widehat{\alpha}_{1,m}^{(1),\mathsf{H}}$ and $\widehat{\alpha}_{1,m}^{(1),\mathsf{E}}$

Explicit computation in the case of the sphere $\mathcal{O}_\delta = \mathcal{B}(0,\delta)$

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$$\mathbf{E}^{\mathrm{sca}}_\delta(\mathbf{x}) \underset{\delta\to 0}{\sim} \delta^3\,\widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4\,\widetilde{\mathbf{E}}_4(\mathbf{x}) + \underset{\delta\to 0}{O}(\delta^5)$$

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$$\widehat{\mathbf{E}}_0(\mathbf{X}) = (\kappa \mathrm{R})^{-3}\{\dots\} + (\kappa \mathrm{R})^{-4}\,\frac{h_{2,-3}}{i}\sum_{m=-2}^{2} \widehat{\alpha}^{(0),\mathrm{H}}_{2,m}\left[-2\,\nabla_{\mathcal{S}^2} Y_{2,m} + 6\,Y_{2,m}\mathbf{e}_r\right] + \underset{\mathrm{R}\to\infty}{O}(\mathrm{R}^{-5})$$

$$\widehat{\mathbf{H}}_0(\mathbf{X}) = (\kappa \mathrm{R})^{-3}\{\dots\} + (\kappa \mathrm{R})^{-4}\,\frac{h_{2,-3}}{i}\sum_{m=-2}^{2} \widehat{\alpha}^{(0),\mathrm{E}}_{2,m}\left[-2\,\nabla_{\mathcal{S}^2} Y_{2,m} + 6\,Y_{2,m}\mathbf{e}_r\right] + \underset{\mathrm{R}\to\infty}{O}(\mathrm{R}^{-5})$$

Boundary conditions on $\widehat{\Gamma} = \mathcal{S}^2 \Longrightarrow$ Unicity of $\widehat{\alpha}^{(0),\mathrm{H}}_{2,m}$ and $\widehat{\alpha}^{(0),\mathrm{E}}_{2,m}$

Outline

Conclusion and perspectives

- These tools are still valid for
  - Extension to **transmission problem**
  - Extension to **multi-scattering problem**

    $\longrightarrow$ Superposition principle
- To finalize
  - **Stability** and **consistance**

    $\longrightarrow$ Far field error

    $\longrightarrow$ Near field error

    $\longrightarrow$ Matching error
  - **Boundary integral equations** for near field problems
- To do
  - Discretization of continuous equations
  - Implementation of the approximation method

¡¡ Thank you for your attention!

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